NORTH-HOLLAND

Products of Conjugacy Classes of Two by Two Matrices*

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ABSTRACT

We compute the covering number and the extended covering number for the group $PSL_2(F)$ with a field F. Similar numbers are defined for any group, and they are computed for the groups $GL_2(F)$, $PGL_2(F)$, and $SL_2(F)$.

1. INTRODUCTION

Let F be a field. Let $GL_2(F)$ be the group of all invertible 2 by 2 matrices over F, let $SL_2(F)$ be the subgroup of all matrices in $GL_2(F)$ whose determinant is one, let $PGL_2(F)$ be the factor group of $GL_2(F)$ modulo its subgroup of scalar matrices, and let $PSL_2(F)$ be the factor group of $SL_2(F)$ modulo its subgroup of scalar matrices. We study products of conjugacy (similarity) classes in the groups $GL_2(F)$, $PGL_2(F)$, $SL_2(F)$, and $PSL_2(F)$. In particular, we are interested in what elements of a group are products of elements from prescribed classes.

For a simple noncommutative group G, its covering number $\mathrm{cn}(G)$ and extended covering number $\mathrm{ecn}(G)$ are defined in [1] as follows. We write that $\mathrm{cn}(G) \leqslant k$ [$\mathrm{ecn}(G) \leqslant k$] if $C^k = G$ [$C_1C_2 \cdots C_k = G$] for every nontrivial conjugacy class C in G [for any nontrivial conjugacy classes C_j in G]. It is

* This research was supported in part by the NSF.

0024-3795/95/$9.50
SSDI 0024-3795(94)00005-X

clear that $\mathrm{cn}(G) \leqslant k$ implies that $\mathrm{cn}(G) \leqslant k + 1$ and that $\mathrm{ecn}(G) \leqslant k$ implies that $\mathrm{cn}(G) \leqslant k$ and $\mathrm{ecn}(G) \leqslant k + 1$. We define $\mathrm{cn}(G)$ [$\mathrm{ecn}(G)$] to be the least k such that $\mathrm{cn}(G) \leqslant k$ [$\mathrm{ecn}(G) \leqslant k$]. It is clear that $2 \leqslant \mathrm{cn}(G) \leqslant \mathrm{ecn}(G)$ for any G. By [1], $\mathrm{ecn}(G) < \infty$ for any finite G.

In [1], $\mathrm{cn}(G)$ was computed for $G = \mathrm{PSL}_2(F)$ with $F = \mathbf{C}$ (the complex numbers) and $F = F_q$ (a finite field of q elements), $q \geqslant 4$. Namely, $\mathrm{cn}(\mathrm{PSL}_2(\mathbf{C})) = 2$ and $\mathrm{cn}(\mathrm{PSL}_2(F_q)) = 3$ [1, Theorem 4.2]. Also $\mathrm{ecn}(\mathrm{PSL}_2(F_q)) = 4$ for any $q \geqslant 4$ by [1, p. 2], and $\mathrm{cn}(\mathrm{PSL}_2(\mathbf{R})) \geqslant 3$ for the reals $\mathbf{R}$ [2, Theorem 2.09].

Note that the group $G = \mathrm{PSL}_2(F)$ is simple if and only if $\mathrm{card}(F) \geqslant 4$. We want to compute $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$ for $G = \mathrm{PSL}_2(F)$, where F is an arbitrary field with $\mathrm{card}(F) \geqslant 4$. Let us first state our answer for $C_1(2)$-fields F. Recall that a field is called a $C_1(2)$-field if every quadratic form in three variables is isotropic or, equivalently, for each quadratic field extension of F every element of F is a norm. Some examples are finite fields and quadratically closed fields. We prove that:

$\mathrm{cn}(G) = 2$ and $\mathrm{ecn}(G) = 3$ when F is quadratically closed, that is, F has no quadratic field extensions or, equivalently, every quadratic form in two variables is isotropic;

$\mathrm{cn}(G) = 3$ and $\mathrm{ecn}(G) = 4$ when F is a $C_1(2)$-field but is not quadratically closed, and either $2F = F$ or every element of F is a square;

$\mathrm{cn}(G) = 4$ and $\mathrm{ecn}(G) = 5$ when F is a $C_1(2)$-field and F has a nonseparable quadratic field extension.

When F is not a $C_1(2)$-field (for those familiar with the notion of the u-invariant [4], this means that $u(F) \geqslant 2$), the answer becomes more complicated (see Theorems 2.4 and 2.5 in the next section). We prove that $\mathrm{ecn}(G) = 4$ when, for each quadratic field extension of F, every element of F is a norm or the negative of a norm, and $\mathrm{ecn}(G) = 5$ otherwise. Concerning $\mathrm{cn}(G)$, we prove that $3 \leqslant \mathrm{cn}(G) \leqslant 4$ [we still consider the case when F is not a $C_1(2)$-field]. The precise conditions for $\mathrm{cn}(G)$ to be 3 are rather complicated (see Remark 9.2 at the end of Section 9). One of these conditions is that every element of F is a sum of two squares. Since this condition does not hold for $F = \mathbf{R}$, for example, $\mathrm{cn}(\mathrm{PSL}_2(\mathbf{R})) = \mathrm{ecn}(\mathrm{PSL}_2(\mathbf{R})) = 4$.

Now let us look at related results from other publications. By [5], in the group $\mathrm{PSL}_n(F)$ with $n \geqslant 3$ and any field F, the product $C_1C_2C_3$ of any three cyclic conjugacy classes C_1, C_2, C_3 contains all nonscalar matrices. When $n = 2$, cyclic means the same as nonscalar, so our Theorems 2.1–2.4 say exactly for which fields the result of [5] holds when $n = 2$, i.e., for which fields F one has $\mathrm{ecn}(\mathrm{PSL}_2(F)) \leqslant 4$.

Products of conjugacy classes in $\mathrm{SL}_n(\mathbf{C})$ were studied in [6] in connection

with hypergeometric functions, Higgs bundles, and variations of Hodge structures. The main result of [6] gives necessary and sufficient conditions on classes $C_1, \ldots, C_k$ for their product to contain all matrices with generic eigenvalues. We obtain in our paper more precise results for arbitrary fields F when $n = 2$.

It is easy to see [7] that $\mathrm{ecn}(\mathrm{PSL}_n(F)) \geqslant n + 1$ for any field F and any $n \geqslant 3$. We conjecture that $\mathrm{cn}(\mathrm{PSL}_n(F)) = \mathrm{ecn}(\mathrm{PSL}_n(F)) - 1 = n$ for any algebraically closed field F. The situation for an arbitrary field is not clear, except it is not difficult to show that $\mathrm{ecn}(\mathrm{PSL}_n(F)) < \infty$.

Going from a field F to other rings requires us to modify the definitions of $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$, which we will do in the next section. [We actually define $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$ for an arbitrary group G.] Now we give a conjecture about $\mathrm{cn}(\mathrm{SL}_n(A))$. A proper version of $\mathrm{SL}_n(A)$ for an arbitrary ring A is the subgroup $\mathrm{E}_n(A)$ of $\mathrm{GL}_n(A)$ generated by elementary matrices (which differ from the identity matrix 1_n in one nondiagonal entry). When A is a field, $\mathrm{E}_n(A) = \mathrm{SL}_n(A)$.

Our conjecture is that $\mathrm{ecn}(\mathrm{E}_n(A)) < \infty$ in the following cases:

$n \geqslant 2$ and A is a commutative local ring;
$n \geqslant 3$ and A is a ring of algebraic numbers;
$n \geqslant 2$ and A is the ring of continuous functions on a finite-dimensional topological space or the ring of smooth functions on a manifold.

By [3], $\mathrm{cn}(\mathrm{E}_n(A)) = \infty$ for all $n \geqslant 2$ when $A = \mathbf{C}[x]$.

2. STATEMENT OF RESULTS

The above definitions of $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$ do not make much sense for a group G which is not simple, because a nontrivial conjugacy class may be contained in a proper normal subgroup, so none of its powers covers G. Another problem arises for a group G which is not perfect, because then any power of C is a single element modulo the commutator subgroup, so it cannot cover G. Therefore, the definitions of $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$ should be changed to be useful for groups which are not simple.

We modify the above definitions of $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$ so that they make sense for an arbitrary group G and in particular for the groups $\mathrm{GL}_2(F)$, $\mathrm{SL}_2(F)$, and $\mathrm{PGL}_2(F)$.

Let $\mathrm{ecn}(G)$ be the least k such that for any conjugacy classes $C_1, \ldots, C_k$ in G such that the normal subgroup of G generated by each of them contains $[G, G]$, the product $C_1 \cdots C_k$ contains every similarity class C_0 such that

$C_0 = C_1 \cdots C_k \bmod [G, G]$. By $\mathrm{cn}(G)$ we denote the least k such that the above condition holds for equal classes $C_1 = \cdots = C_k$.

It is clear that $\mathrm{cn}(G) \leqslant \mathrm{ecn}(G)$. It is also clear that for a simple noncommutative group G, our new definitions agree with the old ones.

Note that the product of conjugacy classes is a conjugacy invariant subset of G which is independent of the ordering of the factors. The condition $C_1 \cdots C_k \supset C_0$ is equivalent to $1 \in C_1 \cdots C_k C_{k+1}$, where $C_{k+1} = C_0^{-1}$.

The normal subgroup structure of $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{PGL}_n(F)$, and $\mathrm{PSL}_n(F)$ is well known. For $G = \mathrm{GL}_n(F)$ with $n \geqslant 2$, the center of the group $\mathrm{GL}_n(F)$ consists of all scalar nonzero matrices; the center of $\mathrm{SL}_n(F)$ consists of scalar matrices of determinant 1; the centers of the groups $\mathrm{PGL}_n(F)$ and $\mathrm{PSL}_n(F)$ are trivial.

We have $[G, G] = \mathrm{SL}_n(F) = [\mathrm{SL}_n(F), \mathrm{SL}_n(F)]$ for $G = \mathrm{GL}_n(F)$, and the group $\mathrm{PSL}_n(F)$ is simple except in the case when $n = 2$ and $\mathrm{card}(F) = 2$ or 3. In particular, the normal subgroup of G generated by a nonscalar matrix contains $[G, G]$ when $G = \mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$ provided that $\mathrm{card}(F) \geqslant 4$ or $n \geqslant 3$. With these two exceptions [namely, $n = 2$ with $\mathrm{card}(F) = 2$ or 3], the condition $C_0 = C_1 \cdots C_n \bmod [G, G]$ is always satisfied when $G = \mathrm{SL}_n(F)$ or $\mathrm{PSL}_n(F)$, and it means $\det C_0 = \det C_1 \cdots \det C_k$ when $G = \mathrm{GL}_n(F)$. So, when $\mathrm{card}(F) \geqslant 4$ or $n \geqslant 3$, we have $\mathrm{cn}(\mathrm{GL}_n(F)) \geqslant \mathrm{cn}(\mathrm{PGL}_n(F))$ and $\mathrm{cn}(\mathrm{SL}_n(F)) \geqslant \mathrm{cn}(\mathrm{PSL}_n(F))$. Also, $\mathrm{ecn}(\mathrm{GL}_n(F)) \geqslant \mathrm{ecn}(\mathrm{PGL}_n(F))$ and $\mathrm{ecn}(\mathrm{SL}_n(F)) \geqslant \mathrm{ecn}(\mathrm{PSL}_n(F))$.

THEOREM 2.1.

(a) *If* $\mathrm{card}(F) = 2$, *then* $\mathrm{GL}_2(F) = \mathrm{SL}_2(F) = \mathrm{PGL}_2(F) = \mathrm{PSL}_2(F)$ *and* $\mathrm{cn}(\mathrm{GL}_2(F)) = \mathrm{ecn}(\mathrm{GL}_2(F)) = 2$.

(b) *If* $\mathrm{card}(F) = 3$, *then* $\mathrm{cn}(\mathrm{PSL}_2(F)) = \mathrm{cn}(\mathrm{SL}_2(F) = \mathrm{ecn}(\mathrm{PSL}_2(F)) = \mathrm{cn}(\mathrm{PGL}_2(F)) = 2$, $\mathrm{ecn}(\mathrm{SL}_2(F)) = \mathrm{cn}(\mathrm{GL}_2(F)) = \mathrm{ecn}(\mathrm{PGL}_2(F)) = 3$, *and* $\mathrm{ecn}(\mathrm{GL}_2(F)) = 4$.

THEOREM 2.2. *If* F *is quadratically closed, then* $\mathrm{cn}(\mathrm{PSL}_2(F)) = \mathrm{cn}(\mathrm{PGL}_2(F)) = 2$, $\mathrm{ecn}(\mathrm{PSL}_2(F)) = \mathrm{ecn}(\mathrm{PGL}_2(F)) = \mathrm{ecn}(\mathrm{SL}_2(F)) = \mathrm{ecn}(\mathrm{GL}_2(F)) = 3$, *and*

$$\mathrm{cn}(\mathrm{SL}_2(F)) = \mathrm{cn}(\mathrm{GL}_2(F)) = \begin{cases} 2 & \text{when } 2F = 0, \\ 3 & \text{when } 2F = F. \end{cases}$$

THEOREM 2.3. *If* F *is a* $C_1(2)$*-field with* $\mathrm{card}(F) \geqslant 4$ *and* F *is not quadratically closed, then* $\mathrm{cn}(G) = \mathrm{ecn}(G) - 1 = 3$ *for* $G = \mathrm{GL}_2(F)$ *and*

$\mathrm{PGL}_2(F)$, *and*

$$\mathrm{cn}(G) = \mathrm{ecn}(G) - 1 = \begin{cases} 3 & \textit{when either } 2F = F \textit{ or} \\ & \textit{every element of } F \textit{ is a square}, \\ 4 & \textit{otherwise} \end{cases}$$

for $G = \mathrm{SL}_2(F)$ *and* $\mathrm{PSL}_2(F)$.

In the case when F is not a $C_1(2)$-field the computation of $\mathrm{cn}(G)$ and $\mathrm{ecn}(G)$ for G becomes more difficult and the answer involves more complicated conditions. In this case, Theorem 2.4 gives the extended covering number of our four groups, while Theorem 2.5 gives our findings for the covering number of these groups.

THEOREM 2.4. *Assume that F is not $C_1(2)$-field. Then:*

(a) $\mathrm{ecn}(G) = 4$ *for* $G = \mathrm{GL}_2(F)$ *and* $G = \mathrm{PGL}_2(F)$;
(b) $\mathrm{ecn}(\mathrm{SL}_2(F)) = 5$;
(c) $\mathrm{ecn}(\mathrm{PSL}_2(F))$

$$= \begin{cases} 4 & \textit{when for each quadratic field extension of } F \textit{ every} \\ & \textit{element of } F \textit{ is a norm or the negative of a norm}, \\ 5 & \textit{otherwise}. \end{cases}$$

Now we introduce the quadratic form $N_t(x, y) = x^2 - txy + y^2$ in x, y, where $t \in F$, which will be used in the statement of the following theorem.

THEOREM 2.5. *Assume that F is not a $C_1(2)$-field. Then:*

(a) $\mathrm{cn}(\mathrm{GL}_2(F))$

$$= \begin{cases} 3 & \textit{when for every } 2 \neq t \in F \textit{ there are } x, y \in F \textit{ such that} \\ & N_t(x, y) = x^2 - txy + y^2 = -1 - t, \\ 4 & \textit{otherwise}. \end{cases}$$

(b) $\mathrm{cn}(\mathrm{PGL}_2(F))$

$$= \begin{cases} 3 & \textit{when for every } t \in F \textit{ either } -1 - t \textit{ or } -1 + t \\ & \textit{has the form } N_t(x, y), \\ 4 & \textit{otherwise}. \end{cases}$$

(c) $3 \leqslant \mathrm{cn}(\mathrm{PSL}_2(F)) \leqslant 4$; *if* $\mathrm{cn}(\mathrm{PSL}_2(F)) = 3$, *then every element of the field F is the sum of two squares and for every element $t \in F$ either* $-1 - t$ *or* $-1 + t$ *has the form* $N_t(x, y)$.

(d) $4 \leqslant \mathrm{cn}(\mathrm{SL}_2(F)) \leqslant 5$; *if* $\mathrm{cn}(\mathrm{SL}_2(F)) = 4$, *then* -1 *in F is a sum of four squares.*

For example, let $F = \mathbf{R}$, which is not a $C_1(2)$-field. Then by Theorem 2.5(a) $\mathrm{cn}(\mathrm{GL}_2(F)) = 4$, by 2.5(b) $\mathrm{cn}(\mathrm{PGL}_2(F)) = 4$, and by 2.5(d) $\mathrm{cn}(\mathrm{SL}_2(F)) = 5$. Now the condition in Theorem 2.5(c) does not hold when $t = 0$ (in fact, it holds if and only if $|t| > 2$, which is more difficult to check), so $\mathrm{cn}(\mathrm{PSL}_2(F)) = 4$. By Theorem 2.4(a), (b) $\mathrm{ecn}(\mathrm{SL}_2(F)) = \mathrm{ecn}(\mathrm{GL}_2(F)) = 5$. Theorem 2.4(a) also gives that $\mathrm{ecn}(\mathrm{PGL}_2(F)) = 4$. The condition in Theorem 2.4(c) holds, so $\mathrm{ecn}(\mathrm{PSL}_2(F)) = 4$.

In the proof of Theorem 2.5, we will use the rigidity of some relations in $\mathrm{GL}_2(F)$. Now we introduce a notion of rigidity for an arbitrary group G (which generalizes that of [6]) and then we state our rigidity result as Theorem 2.6.

A relation $g_1 \cdots g_m = 1$ in G is called *rigid* if for any relation $g_1' \cdots g_m' = 1$ with g_j' similar to g_j for $j = 1, \ldots, m$, there is h in G such that $g_j = hg_j'h^{-1}$.

An interesting problem is to describe all rigid relations in the groups $\mathrm{GL}_n(F)$. In this paper we restrict ourselves to the case $n = 2 = m - 1$.

THEOREM 2.6. *A relation $g_1g_2g_3 = 1$ in $\mathrm{GL}_2(F)$ is rigid if at least one of the g_j has distinct eigenvalues and there is no common eigenvector over F for all g_j.*

3. SIMILARITY CLASSES IN $\mathrm{GL}_2(F)$ AND $\mathrm{SL}_2(F)$

We describe the similarity classes in $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ for an arbitrary field F. Every nonscalar matrix $g \in \mathrm{GL}_2(F)$ is similar to the companion matrix

$$\begin{pmatrix} 0 & -r \\ 1 & t \end{pmatrix}, \tag{3.1}$$

where $t = \mathrm{tr}\, g$ and $r = \det g$ are the only conjugacy invariants of g.

Every matrix $g \in \mathrm{SL}_2(F)$ with distinct eigenvalues in F is also similar in $\mathrm{SL}_2(F)$ to the companion matrix (3.1) with $r = 1$, i.e., to the matrix

$$h_t = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}. \tag{3.2}$$

It is also similar to its inverse g^{-1}.

In general, every nonscalar matrix g in $\mathrm{SL}_2(F)$ is similar in $\mathrm{SL}_2(F)$ to

the matrix

$$\begin{pmatrix} 0 & -1/c \\ c & t \end{pmatrix} \tag{3.3}$$

with $t = \operatorname{tr} g$. The element $c \neq 0$ here is unique up to the determinants, $\det \mathrm{GL}_1(F[g])$, of invertible matrices in the matrix ring $F[g] = F + Fg$. This ring is isomorphic to $F[x]/(x^2 - tx + 1)F[x]$. Note that $\det F[g]$ is the range of the quadratic form $N_t(x, y) = x^2 - txy + y^2 = \det(x1_2 - yg)$.

When the eigenvalues of g are not in F, $F[g]$ is a field extension of F and the determinant, $\det: F[g] \to F$, coincides with the norm. When g has equal eigenvalues in F, $\det F[g]$ consists of all squares in F. When g has distinct eigenvalues in F, $F[g]$ is the direct product of two copies of F and $\det F[g] = F$.

We define the *corner invariant* $\chi(g) = \chi(C)$ as the set of all c in

$$g' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.4}$$

where g' ranges over the conjugacy class C of g in $\mathrm{SL}_2(F)$.

Clearly, $\chi(C^{-1}) = -\chi(C)$ while $\operatorname{tr} C^{-1} = \operatorname{tr} C$. Note also that $\chi(C) = 0$ if and only if C is scalar and that $0 \in \chi(C)$ if and only if C has eigenvalues in F. If $0 \notin \chi(C)$, then $\chi(C) = c \det \mathrm{GL}_1(F[g]) = -b \det \mathrm{GL}_1(F[g])$ is a coset in $\mathrm{GL}_1(F)$, where $g \in C$ and b, c are as in (3.4) with any $g' \in C$. Here $\det \mathrm{GL}_1(F[g])$ coincides with $\det h$ such that $hgh^{-1} = h_t$, where $t = \operatorname{tr} g$.

A nonscalar $g \in \mathrm{SL}_2(F)$ has equal eigenvalues if and only if it is similar to (3.3) with $c \neq 0$ and $t = \pm 2$. The element $c \neq 0$ here is unique up to multiplication by a nonzero square in F. Every nonscalar matrix g in $\mathrm{SL}_2(F)$ with $\operatorname{tr} g = 2$ has the form

$$\begin{pmatrix} 1 + uvc & cv^2 \\ -cu^2 & 1 - uvc \end{pmatrix} \tag{3.5}$$

with $(u, v) \neq (0, 0)$, $u, v \in F$. When we pass from g to g^{-1}, c is replaced by $-c$.

In our proofs of Theorems 2.3 and 2.4, we will use the following result.

LEMMA 3.6. *If F'/F is a separable quadratic field extension, then there is an element $\mu \in F'/F$ of norm 1.*

Proof. Let $F' = F[z]$ with $z^2 - tz + r = 0$, $r, t \in F$. Since the extension is separable, we can assume that $t \neq 0$. Then the norm of $\mu = 1 - zt/r \in F' \setminus F$ is $1^2 - t^2/r + t^2/r = 1$. ∎

4. PROOF OF THEOREM 2.1

(a): Let $\operatorname{card}(F) = 2$. The group $\mathrm{GL}_2(F) = \mathrm{SL}_2(F) = \mathrm{PGL}_2(F) = \mathrm{PSL}_2(F)$ is isomorphic to the symmetric group $G = \mathrm{S}_2 = C_1 \cup C_2 \cup C_3$ with C_j consisting of all elements of order j in G. We have $C_2C_2 = C_3C_3 = C_1 \cup C_3 = \mathrm{A}_3 = [\mathrm{S}_3, \mathrm{S}_3]$ and $C_2C_3 = C_3C_2 = C_2 = \mathrm{S}_3 \setminus \mathrm{A}_3$. So $\operatorname{cn}(G) = \operatorname{ecn}(G) = 2$.

(b): Let $\operatorname{card}(F) = 3$. The group $\mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/\{\pm 1_2\}$ is isomorphic to the alternating group $G = \mathrm{A}_4 = C_1 \cup C_2 \cup C_3 \cup C_3^{-1}$, where C_j is the conjugacy class of an element of order j in G. We have $C_2C_2 = C_1 \cup C_2 = [G, G] \neq G$, $C_3C_3^{-1} = G$, and $C_2C_3 = C_3$. So $\operatorname{cn}(G) = \operatorname{ecn}(G) = 2$.

The conjugacy classes $\pm C_1, C_2, \pm C_3, \pm C_3^{-1}$ in $\mathrm{SL}_2(F)$ [our notation for the classes corresponds to that for the conjugacy classes in $\mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/\{\pm 1_2\}$] are represented by the matrices $\pm 1_2, h_0, \pm h_2, \pm h_2^{-1}$, where h_t is the companion matrix with trace t [see (3.2)]. The orders of elements in $C_1, -C_1, C_2 = -C_2 = C_2^{-1}, C_3, -C_3, C_3^{-1}, -C_3^{-1}$ are 1, 2, 4, 3, 6, 3, 6 respectively.

We have

$$C_2C_2 = C_1 \cup -C_1 \cup C_2 = [G, G],$$

$$C_3C_3 = C_3^{-1} \cup -C_3^{-1},$$

$$C_2C_3 = C_3 \cup -C_3,$$

$$C_2C_3^{-1} = C_3^{-1} \cup -C_3^{-1},$$

$$C_3C_3^{-1} = C_1 \cup C_2 \neq [G, G],$$

$$C_3C_3C_3 = [G, G].$$

So $\operatorname{cn}(\mathrm{SL}_2(F)) = 2$ and $\operatorname{ecn}(\mathrm{SL}_2(F)) = 3$.

Consider now $G = \mathrm{GL}_2(F)$. In addition to the conjugacy classes $\pm C_1, C_2, \pm C_3$ of the above matrices $\pm 1_2, h_0, \pm h_2$ of determinant 1 (note that now $C_3^{-1} = C_3$), we have the conjugacy classes $T_0 = T_0^{-1}$ and $\pm T_1$ with $\det T_0 = -1 = \det T_1$ and $\operatorname{tr} T_0 = 0$, $\operatorname{tr} T_1 = 1$, $T_1^{-1} = -T_1$. The classes

$\pm C_1$ and C_2 generate proper subgroups of $[G, G] = \mathrm{SL}_2(F)$. Here is the multiplication table for the other classes:

$$C_3C_3 = C_3 \cup -C_3 \cup C_2 \cup C_1 = SL_2(F) \setminus -C_1,$$

$$T_0T_1 = C_2 \cup C_3 \cup -C_3,$$

$$T_0T_0 = \mathrm{SL}_2(F),$$

$$T_1T_1 = -C_1 \cup C_2 \cup C_3,$$

$$T_0C_3 = T_1 \cup -T_1 \cup T_0,$$

$$T_1C_3 = T_0 \cup -T_1.$$

Since T_1T_1 does not contain C_1, $\mathrm{cn}(\mathrm{GL}_2(F)) \geqslant 3$. Since in addition T_0T_1 does not contain $\pm C_1$, $\mathrm{ecn}(\mathrm{PGL}_2(F)) \geqslant 3$. Since $C_3T_1T_1$ does not contain $-C_1$, $\mathrm{ecn}(\mathrm{GL}_2(F)) \geqslant 4$.

On the other hand, $C_3C_3C_3 = \mathrm{SL}_2(F)$ and $T_iT_iT_i = \mathrm{GL}_2(F) \setminus \mathrm{SL}_2(F)$, hence $\mathrm{cn}(\mathrm{GL}_2(F)) \leqslant 3$. Also $\mathrm{ecn}(\mathrm{PGL}_2(F)) \leqslant 3$ and $\mathrm{ecn}(\mathrm{GL}_2(F)) \leqslant 4$ by direct computations.

5. PROOF OF THEOREM 2.2

LEMMA 5.1. *Let C_1, C_2, C_0 be nonscalar conjugacy classes in $\mathrm{GL}_2(F)$, and let $\det C_1 \det C_2 = \det C_0$. Assume that the eigenvalues of C_2 are in F and either they are distinct, or $C_1 \neq \mu C_0$ for an eigenvalue μ of C_2, or the eigenvalues of C_1 are in F and $\mathrm{card}(F) \neq 2$. Then $C_1C_2 \supset C_0$.*

Proof. We choose the companion matrix

$$g_1 = \begin{pmatrix} 0 & r_1 \\ 1 & t_1 \end{pmatrix}$$

in C_1 and an upper triangular matrix

$$g_2 = \begin{pmatrix} \lambda & b \\ 0 & \mu \end{pmatrix}$$

in C_2. Then $\mathrm{tr}\, g_1g_2 = b + \mu t_1$.

When $\lambda \neq \mu$, b is an arbitrary element of F, so $\operatorname{tr} g_1 g_2 = \operatorname{tr} C_0$ for some b; hence $g_1 g_2 \in C_0$, and we are done.

When $\lambda = \mu$, b is an arbitrary nonzero element of F, so we are also done unless $\operatorname{tr} C_0 = \mu t_1$.

It remains to consider the case when $\lambda = \mu$, $\operatorname{tr} C_0 = \mu t_1$, the eigenvalues of C_1 are in F, and $\operatorname{card} F \neq 2$. In this case we replace g_1 by

$$g_1 = \begin{pmatrix} \lambda' & b' \\ 0 & \mu' \end{pmatrix} \in C_1.$$

Then

$$g_1 g_2 = \begin{pmatrix} \lambda'\lambda & \lambda' b + b'\lambda \\ 0 & \mu'\mu \end{pmatrix}.$$

Since $\operatorname{card}(F) \neq 2$ and b is an arbitrary nonzero element of F, we can arrange that $\lambda' b + b'\lambda \neq 0$; hence $g_1 g_2 \in C_0$, and we are done. ■

COROLLARY 5.2. *Assume that the field F is quadratically closed. Then the product $C_1 C_2$ of any two nonscalar conjugacy classes in $\mathrm{SL}_2(F)$ (in $\mathrm{GL}_2(F)$) contains all nonscalar matrices (all nonscalar matrices with determinant equal to $\det C_1 \det C_2$). Therefore* $\operatorname{ecn}(G) \leqslant 3$ *for* $G = \mathrm{GL}_2(F)$, $\mathrm{SL}_2(F)$, $\mathrm{PGL}_2(F)$, $\mathrm{PSL}_2(F)$.

Now we can complete our proof of Theorem 2.2. Assume that F is quadratically closed. Note that the group $\mathrm{SL}_2(F)$ contains two conjugacy classes C_1, C_2 which are not the same in $\mathrm{PSL}_2(F)$. For example, C_1 and C_2 are the classes of the companion matrices with traces 0 and 1 respectively. Then $C_1 C_2^{-1}$ does not contain any scalar matrices; hence $\operatorname{ecn}(G) \geqslant 3$ for $G = \mathrm{GL}_2(F)$, $\mathrm{SL}_2(F)$, $\mathrm{PGL}_2(F)$, $\mathrm{PSL}_2(F)$. By Corollary 5.2, the inverse inequalities hold, so $\operatorname{ecn}(G) = 3$ for these G.

On the other hand, in the group $\mathrm{GL}_2(F)$, CC contains the scalar matrix $(\det C)1_2$ for any conjugacy class C. So, using Lemma 5.1, $\operatorname{cn}(G) = 2$ for $G = \mathrm{PGL}_2(F)$ and $G = \mathrm{PSL}_2(F)$.

But CC does not contain $-(\det C)1_2$, which has the same determinant, $(\det C)^2$, as $(\det C)1_2$, if $\operatorname{tr} C \neq -\operatorname{tr} C$. So $\operatorname{cn}(G) \geqslant 3$ for $G = \mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ when $-1 \neq 1$ in F (i.e., $2F = F$).

6. PROOF OF THEOREM 2.3

LEMMA 6.1. *Assume that* $\operatorname{card}(F) \neq 2, 3$. *Let C_1, C_2, C_3, C_0 be nonscalar conjugacy classes in $\mathrm{GL}_2(F)$ such that* $\det C_0 = \det C_1 \det C_2 \det C_3$.

Then $C_1C_3C_3 \supset C_0$. *Therefore* $\mathrm{cn}(\mathrm{PGL}_2(F)) \leqslant \mathrm{cn}(\mathrm{GL}_2(F)) \leqslant \mathrm{ecn}(\mathrm{GL}_2(F)) \leqslant 4$.

Proof. Set $r = \det C_1 \det C_2$. We pick $0 \neq \lambda \in F$ such that $\lambda^2 \neq r$. By Lemma 5.1 both C_1C_2 and $C_0C_3^{-1}$ contain

$$\begin{pmatrix} \lambda & 0 \\ 0 & r/\lambda \end{pmatrix}.$$

So $C_1C_2C_3 \supset C_0$. ∎

LEMMA 6.2. *Let* C_1, C_2 *be conjugacy classes in* $\mathrm{SL}_2(F)$ *with eigenvalues outside of* F *and* $\pm 1 \neq \lambda \in F$. *Then the diagonal matrix* $\mathrm{diag}(\lambda, \lambda^{-1}) \in C_1C_2$ *if and only if* $-\lambda \in \chi(C_1)\chi(C_2)$.

Proof. Let $\mathrm{diag}(\lambda, \lambda^{-1}) = g_1g_2$ with $g_j \in C_j$ for $j = 1, 2$. Write

$$g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

The equality $g_1^{-1}\,\mathrm{diag}(\lambda, \lambda^{-1}) = g_2$ gives that $-c_1\lambda = c_2$; hence $-\lambda = c_1^{-1}c_2 \in \chi(C_1)\chi(c_2)$.

Conversely, assume that $-\lambda \in \chi(C_1)\chi(C_2)$. Then we write $-\lambda = c_1^{-1}c_2$ with $c_j \in \chi(C_j)$ for $j = 1, 2$. We set

$$g_1 = \begin{pmatrix} t_1 & \dfrac{-1}{c_1} \\ c_1 & 0 \end{pmatrix} \in C_1 \quad \text{and} \quad g_0 = \begin{pmatrix} \lambda & \dfrac{t_1/\lambda - t_2}{c_1} \\ 0 & \dfrac{1}{\lambda} \end{pmatrix} \in \mathrm{SL}_2(F)$$

with $t_j = \mathrm{tr}\, C_j$. Then

$$g_1^{-1}g_0 = \begin{pmatrix} 0 & \dfrac{1}{c_1} \\ -c_1 & t_1 \end{pmatrix} \begin{pmatrix} \lambda & \dfrac{t_1/\lambda - t_2}{c_1} \\ 0 & \dfrac{1}{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{c_2} \\ -c_2 & t_2 \end{pmatrix} \in C_2;$$

hence $g_0 \in C_1C_2$. Since g_0 is similar to $\mathrm{diag}(\lambda, \lambda^{-1})$, we conclude that $\mathrm{diag}(\lambda, \lambda^{-1}) \in C_1C_2$. ∎

COROLLARY 6.3. *Assume that F is a $C_1(2)$-field with* $\mathrm{card}(F) \neq 2, 3$. *Let C_1, C_2, C_3, C_0 be nonscalar conjugacy classes in $\mathrm{SL}_2(F)$. Assume that either $(\mathrm{tr}\, C_j)^2 \neq 4$ for some $j = 0, 1, 2, 3$ or $2F = F$. Then $C_1C_2C_3 \supset C_0$.*

Proof. Suppose first that $(\mathrm{tr}\, C_j)^2 = 4$ for all j. By the condition of the corollary, $2F = F$. For each $j = 1, 2, 3$, replacing, if necessary, both C_j and C_0 by $-C_j$ and $-C_0$ respectively, we can assume that $\mathrm{tr}\, C_j = 2$ for $j = 1, 2, 3$. Pick a nonzero $b_j \in \chi(C_j)$.

If $\mathrm{tr}\, g_0 = 2$, we find nonzero $x_1, x_2 \in F$ such that $b_1x_1^2 + b_2x_2^2 = b_0 - b_3$ [using that F is a $C_1(2)$-field]. Then

$$g_j = \begin{pmatrix} 1 & 0 \\ b_jx_j^2 & 1 \end{pmatrix} \in C_j$$

for $j = 0, 1, 2, 3$, where $x_j = 1$ for $j = 0, 3$ and $g_1g_2g_3 = g_0$.

Assume now that $\mathrm{tr}\, g_0 = -2$. Then we find nonzero $x, y \in F$ such that $b_1b_2x^2 + b_0b_3y^2 = 4$. Set

$$g_1 = \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \in C_1, \qquad g_2 = \begin{pmatrix} 1 & -b_2x^2 \\ 0 & 1 \end{pmatrix} \in C_2,$$

$$g_3 = \begin{pmatrix} 1 & -b_3y^2 \\ 0 & 1 \end{pmatrix} \in C_3, \quad \text{and} \quad g_0 = \begin{pmatrix} 1 & 0 \\ b_0 & 1 \end{pmatrix} \in C_0.$$

Then $\mathrm{tr}\, g_1g_2 = 2 - b_1b_2x^2 = 2 + b_0b_3y^2 = \mathrm{tr}\, g_0g_3^{-1} \neq 2$; hence g_1g_2 and $g_0g_3^{-1}$ are similar in $\mathrm{SL}_2(F)$. So $C_1C_2C_3 \supset C_0$.

Assume that $(\mathrm{tr}\, C_j)^2 \neq 4$ for some $j = 0, 1, 2, 3$. Without loss of generality, we can assume that $(\mathrm{tr}\, C_0)^2 \neq 4$. The condition that F is a $C_1(2)$-field implies $\chi(C_0) = F$. By Lemmas 5.1 and 6.2, C_1C_2 contains the diagonal matrix $g = \mathrm{diag}(\lambda, 1/\lambda)$ with some $\lambda \neq \pm 1$ in F. Using Lemma 6.2 again, we obtain that $g \in C_0C_3^{-1}$. So $C_1C_2C_3 \supset C_0$. ∎

LEMMA 6.4. *Let C be a conjugacy class in $\mathrm{GL}_2(F)$ with eigenvalues not in F. Then $g_0 \notin CC$, where*

$$g_0 = \begin{pmatrix} d & 1 \\ 0 & d \end{pmatrix} \qquad \textit{with} \quad d = \det C.$$

If in addition either $\operatorname{tr} C = 0$ *or* -1 *is a square in* F, *then* CC *does not contain the matrix* $-g_0$.

Proof. Assume that $g_1 g_2 = g_0$ with $g_1, g_2 \in G$. Then $g_1^{-1} g_0 = g_2$. So $\operatorname{tr} g_0^{-1} g_1 \neq \operatorname{tr} g_1 = \operatorname{tr} g_2$. When $\operatorname{tr} C = 0$, we obtain a contradiction also from $g_0^{-1} g_1 = -g_2$, because then $\operatorname{tr} g_0^{-1} g_1 \neq \operatorname{tr} g_1 = 0 = \operatorname{tr}(-g_2)$. When $-1 = \varepsilon^2$ in F, the eigenvalues of εC are not in F, so $g_0 \notin \varepsilon C = -CC$; hence $-g_0 \notin CC$.

COROLLARY 6.5. *If* F *has a quadratic field extension and* $\operatorname{card}(F) \geqslant 4$, *then* $\operatorname{cn}(\mathrm{PGL}_2(F)) \geqslant 3$, $\operatorname{cn}(\mathrm{PSL}_2(F)) \geqslant 3$, $\operatorname{ecn}(\mathrm{PGL}_2(F)) \geqslant 4$, *and* $\operatorname{ecn}(\mathrm{PSL}_2(F)) \geqslant 4$. *If in addition* F *has a nonseparable quadratic field extension, then* $\operatorname{ecn}(\mathrm{SL}_2 F) \geqslant 5$.

Proof. We will show that there are nonscalar matrices g_0 and g in $\mathrm{SL}_2(F)$ such that $g_0, -g_0 \notin CC$, where C is the conjugacy class of g in $\mathrm{GL}_2(F)$. This will imply that $\operatorname{cn}(\mathrm{PGL}_2(F)) \geqslant 3$, $\operatorname{cn}(\mathrm{PSL}_2(F)) \geqslant 3$, and [since $\pm 1_2 \notin CCC_0^{-1}$, where C_0 is the conjugacy class of g_0 in $\mathrm{GL}_2(F)$] $\operatorname{ecn}(\mathrm{PGL}_2(F)) \geqslant 4$, $\operatorname{ecn}(\mathrm{PSL}_2(F)) \geqslant 4$.

Case 1: -1 *is not a square in* F. Then, by Lemma 6.4, $g_0, -g_0 \notin CC$, where

$$g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and C is the conjugacy class of

$$g = h_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in $\mathrm{GL}_2(F)$.

Case 2: -1 *is a square in* F, *and* F *has a separable quadratic field extension.* By Lemma 3.6 there is $t \in F$ such that the matrix h_t [see (3.2)] has eigenvalues outside of F. Let C be the conjugacy class of $g = h_t \in \mathrm{SL}_2(F)$. By Lemma 6.4,

$$g_0, -g_0 \notin CC, \qquad \text{where} \quad g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

So $\operatorname{cn}(\mathrm{PGL}_2(F)) \geqslant 3$ and $\operatorname{cn}(\mathrm{PSL}_2(F)) \geqslant 3$.

Case 3: F has a nonseparable quadratic field extension. Then $2F = 0$, and there is $b \in F$ which is not a square. Since $\chi(g_0)$ has no nonzero squares, $-g_0 = g_0 \notin CC$, where

$$g_0 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

and C is the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}_2(F) = \mathrm{PSL}_2(F)$.

So $g_0, -g_0 \notin CC$ in all cases.

Now we have to prove that $\mathrm{ecn}(\mathrm{SL}_2(F)) \geqslant 5$ under the assumption of case 3. Let b, C and g_0 be as in case 3, and let C_0 be the conjugacy class of g_0 in $\mathrm{SL}_2(F)$. We will show that $1_2 \notin CCCC_0$. Suppose that $g_1 g_2 = g_0 g_3$ where $g_1, g_2, g_3 \in C = C^{-1}$. Taking traces, we have $2 + x^2 = \mathrm{tr}\, g_1 g_2 = \mathrm{tr}\, g_0 g_3 = 2 + by^2$; hence $\mathrm{tr}\, g_1 g_2 = 2 = \mathrm{tr}\, g_0 g_3$. Now $\chi(g_1 g_2)$ consists of all squares in F, while $\chi(g_0 g_3)$ contains $b + y^2$ for some $y \in F$. Since $2F = 0$ and b is not a square, we obtain a contradiction. ∎

LEMMA 6.6. *Let C be a nonscalar conjugacy class in $\mathrm{GL}_2(F)$ with $\det C = 1$ and $t = \mathrm{tr}\, C \neq \pm 2$. Then the set $\mathrm{tr}\, CC$ consists of all $t_0 \in F$ such that the equation $x^2 - txy + y^2 = -(t_0 - 2)(t^2 - t_0 - 2)$ has a solution $x, y \in F$.*

Proof. If C has distinct eigenvalues in F, then $CC = \mathrm{SL}_2(F)$ by Lemma 5.1, and the quadratic form $x^2 - txy + y^2$ represents everything in F. Assume now that the eigenvalues of C are not in F, i.e., the characteristic polynomial $\det(g - z1_2) = z^2 - tz + 1$ of C is irreducible, where $g \in C$.

Every matrix in CC is similar to a matrix of the form

$$\begin{pmatrix} t - d & b \\ -[1 + d(d - t)]/b & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

whose trace is $t_0 = b + [1 + d(d - t)]/b + dt$. We multiply this equation by b and take all terms to the left hand side to obtain

$$b^2 + d^2 + tbd - td - t_0 b + 1 = 0. \tag{6.7}$$

To get rid of the linear terms, we substitute

$$b = u + \frac{2t_0 - t^2}{4 - t^2} \quad \text{and} \quad d = v + \frac{t(t_0 - 2)}{4 - t^2}$$

in (6.7) and obtain

$$\det(ug + v1_2) = u^2 + tuv + v^2 = \frac{(t_0 - 2)(t^2 - t_0 - 2)}{t^2 - 4}.$$

Now we write $(ug + v1_2)(g^2 - 1_2) = xg - y1_2$ with $x, y \in F$ to obtain

$$x^2 - txy + y^2 = \det(xg - y1_2) = \det(ug + v1_2)\det(g^2 - 1_2)$$

$$= -(t_0 - 2)(t^2 - t_0 - 2). \qquad \blacksquare$$

Now we can complete our proof of Theorem 2.3. Assume that F is a $C_1(2)$-field with $\operatorname{card}(F) \geqslant 4$ and F is not quadratically closed. The trace is the only invariant of a nonscalar matrix in $\mathrm{SL}_2(F)$.

Let C be a nonscalar conjugacy class in $G = \mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$. By Lemmas 6.1 and 6.2, CCC contains all nonscalar matrices g_0 with $\det g = (\det C)^3$. Let us show that CCC also contains scalar matrices $g_0 = \lambda 1_2$ with $0 \neq \lambda \in F$ and $\det g_0 = (\det C)^3 = \lambda^2$.

We have $\det C = \mu^2$ with $\mu = \lambda/\det C$. The inclusion $g_0 \in CCC$ we want to prove takes the form $1_2 \in C'C'C'$ with $C' = C/\mu$. Since $\det C' = 1$, we have $C'^{-1} = C'$, so we have to prove that $C' \subset C'C'$.

Set $t = -\operatorname{tr} C'$. If $t = \pm 2$, then $C' \subset C'C'$ by Lemma 5.1. Otherwise, $C' \subset C'C'$ by Lemma 6.5. So $\operatorname{cn}(G) \leqslant 3$ for $G = \mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$. Combining this with Corollary 6.4, we obtain that $\operatorname{cn}(G) = 3$ for $G = \mathrm{SL}_2(F)$, $\mathrm{GL}_2(F)$, $\mathrm{PSL}_2(F)$, $\mathrm{PGL}_2(F)$.

Since $\operatorname{cn}(\mathrm{PSL}_2(F)) \geqslant 3$, there is a nonscalar conjugacy class C in $\mathrm{SL}_2(F)$ such that CC does not contain C_0, $-C_0$ for a conjugacy class C_0 in $\mathrm{SL}_2(F)$. Since $C = C^{-1}$, C_0 is not scalar. Therefore $\operatorname{ecn}(\mathrm{PSL}_2(F)) \geqslant 4$. It follows that $\operatorname{ecn}(G) \geqslant 4$ for $G = \mathrm{SL}_2(F)$, $\mathrm{GL}_2(F)$, $\mathrm{PSL}_2(F)$, $\mathrm{PGL}_2(F)$. On the other hand, Lemma 6.1 gives the inverse inequalities. So $\operatorname{ecn}(G) = 4$ for $G = \mathrm{SL}_2(F)$, $\mathrm{GL}_2(F)$, $\mathrm{PSL}_2(F)$, $\mathrm{PGL}_2(F)$.

7. PROOF OF THEOREM 2.4

LEMMA 7.1. *Assume that* $\operatorname{card}(F) \geqslant 4$. *Let* C_1, C_2 *be nonscalar conjugacy classes in* $\mathrm{SL}_2(F)$. *Then* C_1C_2 *contains a nonscalar diagonal matrix.*

Proof. We choose nonzero $c_j \in \chi(C_j)$ such that $c_1 \neq \pm c_2$. Set $\lambda = -c_2/c_1 \neq \pm 1$, and choose

$$g_1 = \begin{pmatrix} a_1 & * \\ c_1 & d_1 \end{pmatrix} \in C_1.$$

Determine $x \in F$ from

$$a_1\lambda + \frac{d_1}{\lambda} + c_1 x = \operatorname{tr} C_2.$$

Set

$$g = \begin{pmatrix} \lambda & x \\ 0 & 1/\lambda \end{pmatrix}.$$

Then $\operatorname{tr} g_1 g = \operatorname{tr} g_2$ and $-c_2 \in \chi(g_1 g)$; hence $g_1 g \in C_2^{-1}$. So $g \in C_1C_2$; hence $\operatorname{diag}(\lambda, 1/\lambda) \in C_1C_2$, because this diagonal matrix is similar to g in $\mathrm{SL}_2(F)$.

LEMMA 7.2. *Assume that* $\operatorname{card}(F) \geqslant 4$. *Let* C_1, C_2, C_3, C_4, C_0 *be nonscalar conjugacy classes in* $\mathrm{SL}_2(F)$. *Then* $C_0 \subset C_1C_2C_3C_4$.

Proof. By Lemma 7.1, C_1C_2 contains the conjugacy class C_5 of $g_5 = \operatorname{diag}(-\lambda, -1/\lambda)$ with some $\lambda \neq \pm 1$ in F. Similarly, C_3C_4 contains the conjugacy class C_6 of $g_6 = \operatorname{diag}(-\mu, -1/\mu)$ with some $\mu \neq \pm 1$ in F.

Note that $\chi(g_5) = \chi(g_6) = F$, i.e., the similarity class of g_j in $\mathrm{SL}_2(F)$ and in $\mathrm{GL}_2(F)$ is one and the same for $j = 5, 6$. By Lemma 5.1 $C_0 \subset C_5C_6$; hence $C_0 \subset C_5C_6 \subset C_1C_2C_3C_4$. ∎

Now we are ready to complete our proof of Theorem 2.4. Assume that F is not a $C_1(2)$-field.

(a): By Corollary 6.5, $\operatorname{ecn}(G) \geqslant 4$ for $G = \mathrm{GL}_2(F), \mathrm{PGL}_2(F)$. By Lemma 6.1, $\operatorname{ecn}(G) \leqslant 4$ for $G = \mathrm{GL}_2(F), \mathrm{PGL}_2(F)$. So $\operatorname{ecn}(G) = 4$ for $G = \mathrm{GL}_2(F), \mathrm{PGL}_2(F)$.

(b): By Lemma 7.2, $\operatorname{ecn}(SL_2(F)) \leqslant 5$. Let us show the inverse inequality.

Let F'/F be a quadratic field extension, and let $c \in F$ not be a norm. If the extension is not separable, then $\operatorname{ecn}(SL_2(F)) \geqslant 5$ by Corollary 6.5. So we assume that F' is separable. By Lemma 3.6, there is an element $z \in F' \setminus F$ of norm 1. Let t be the trace of z. Note that $t \neq \pm 2$. The norm of $xz - y$ is the quadratic form $x^2 - txy + y^2$ which does not represent c. Let $C_1 = C_4$ be the conjugacy class of h_2 [see (3.2)] in $SL_2(F)$. Let C_2 be the conjugacy class of

$$\begin{pmatrix} 1 & 1/(t+2) \\ 0 & 1 \end{pmatrix}$$

in $SL_2(F)$, and $-C_3$ the conjugacy class of (3.3) in $SL_2(F)$.

We claim that $1_2 \notin C_1C_2C_3C_4$. Otherwise we have $g_1g_2g_3g_4 = 1_2$ with $g_j \in C_j$, or $g_1^{-1}g_2^{-1} = g_3g_4$. Without loss of generality, we can assume that

$$g_4 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

The matrix $g_1^{-1}g_2^{-1}$ is similar to

$$\begin{pmatrix} 1-uv & v^2 \\ -u^2 & 1+uv \end{pmatrix}\begin{pmatrix} 1 & -1/(t+2) \\ 0 & 1 \end{pmatrix}$$

with $u, v \in F$, so its trace is $2 + u^2/(t+2)$. On the other hand, the trace of g_3g_4 has the form $t + cw$ with a nonzero norm $w \in F$. From $2 + u^2/(t+2) = t + cw$ we obtain that $cw(t+2) = u^2 - (t^2 - 4)$.

Note that $t + 2$ is a norm (take $x = 1 = -y$ in $x^2 - txy + y^2$) and so is $u^2 - (t^2 - 4)$ (take $x = (u+t)/2$, $y = 1$). Therefore c is a norm, which is a contradiction.

(c): By Lemma 7.2, $\operatorname{ecn}(PSL_2(F)) \leqslant \operatorname{ecn}(SL_2(F)) \leqslant 5$. By Corollary 6.5, $\operatorname{ecn}(PSL_2(F)) \geqslant 4$.

Assume now that for each quadratic field extension of F every element of F is a norm or the negative of a norm, and let us prove that $\operatorname{ecn}(PSL_2(F)) \leqslant 4$. Let C_j be conjugacy classes in $SL_2(F)$ for $j = 0, 1, 2, 3, 4$ such that the C_j are not scalar for $j \geqslant 1$. We have to prove that C_0 or $-C_0 \subset C_1C_2C_3C_4$. If C_0 is not scalar, $C_0 \subset C_1C_2C_3C_4$ by Lemma 7.2. Assume now that $C_0 = 1_2$. By Lemma 7.1, C_1C_2 contains the conjugacy class $C_5 = C_5^{-1}$ of $g_5 = \operatorname{diag}(-\lambda, -1/\lambda)$ with some $\lambda \neq \pm 1$ in F. By Lemmas 5.1 and 6.2, C_3C_4 contains g_5 or $-g_5$. So $C_1C_2C_3C_4$ contains 1_2 or -1_2 respectively.

8. PROOF OF THEOREM 2.6

Let $g_1g_2g_3 = 1 = g_1'g_2'g_3'$ with g_j' similar to g_j for $j = 1, 2, 3$ and let one of the g_j have distinct eigenvalues. Assume also that there is no common eigenvector over F for all g_j. We have to prove that there is h in G such that $g_j = hg_j'h^{-1}$ for all j.

Without loss of generality, we can assume that $g_2 = g_2'$ has distinct eigenvalues. Consider first the case when the eigenvalues belong to F. Then we can assume that $g_2 = g_2' = \operatorname{diag}(\lambda, \mu)$ is a diagonal matrix. We write

$$g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad g_1' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

We have

$$a + d = \operatorname{tr} g_1 = a' + d' = \operatorname{tr} g_1',$$

$$a\lambda + d\mu = \operatorname{tr} g_3^{-1}, \quad \text{and} \quad a'\lambda + d'\mu = \operatorname{tr} g_3^{-1} = \operatorname{tr} g_3'^{-1}.$$

Since $\lambda \neq \mu$, we conclude that $a = a'$ and $d = d'$.

If $b = 0$ or $c = 0$, then there is a common eigenvector over F for all g_j [namely, $(1, 0)$ or $(0, 1)$]. Otherwise, $c' \neq 0$ and $g_1 = hg_1'h^{-1}$ with $h = \operatorname{diag}(1/c, 1/c')$. Since h commutes with $g_2 = g_2'$, we have $g_j = hg_j'h^{-1}$ for $j = 1, 2, 3$.

Note that without the condition about the common eigenvector it may happen that $bc = 0$. In this case, $b'c' = 0$. Unless the zero entries of g_1 and g_1' match exactly, there is no h in G such that $g_j = hg_j'h^{-1}$ for $j = 1, 2$. So the condition is necessary.

Assume now that the eigenvalues of g_2 are outside of F. A quadratic field extension F' of F contains the eigenvalues λ, μ. So there is a matrix $h \in \mathrm{GL}_2(F')$ such that $g_j = hg_j'h^{-1}$ for $j = 1, 2, 3$ provided that there is no common eigenvector over F' for all g_j.

If there is such a vector, then by applying the nontrivial automorphism of F'/F which switches λ and μ, we see that the matrices g_j are simultaneously diagonalizable over F'. So $g_1 = \operatorname{diag}(a, d)$. Since $\operatorname{tr} g_1 = \operatorname{tr} g_1'$ and $\operatorname{tr} g_3^{-1} = \operatorname{tr} g_1g_2 = \operatorname{tr} g_1'g_2 = \operatorname{tr}(g_3'^{-1})$, where $g_2 = \operatorname{diag}(\lambda, \mu)$, we conclude that the diagonal entries of g_1' are a, d. Since $\det g_1 = \det g_1'$, we conclude that g_1' is upper or lower triangular. Therefore g_1' and $g_2 = g_2'$ has a common eigenvector over F'. So they are simultaneously diagonalizable over F'; hence $g_1 = g_1'$. Thus, there is a matrix $h \in \mathrm{GL}_2(F')$ such that $g_j = hg_j'h^{-1}$ for $j = 1, 2, 3$.

We write $h = f + \lambda f'$ with 2 by 2 matrices f, f' over F. If $f \in \mathrm{GL}_2(F)$, then $g_j = fg_j'f^{-1}$ for $j = 1, 2, 3$ and we are done. If $f' \in \mathrm{GL}_2(F)$ (e.g., $f = 0$), then $g_j = f'g_j'f'^{-1}$ for $j = 1, 2, 3$ and we are done. Assume now that $\det f = 0$ and $f \neq 0$. Conjugating g_1, g_2, g_3, h by a matrix in $\mathrm{GL}_2(F)$, we can assume that the second column of f is zero. Then, since $fg_j' = g_jf$, we obtain that all g_j are lower triangular, hence they have a common eigenvector over F, which contradicts the condition of the theorem.

9. PROOF OF THEOREM 2.5

(a): Assume that the equation $x^2 - txy + y^2 = -t - 1$ has a solution $x_0, y_0 \in F$ for every $2 \neq t \in F$. We want to prove that $\mathrm{cn}(\mathrm{GL}_2(F)) \leqslant 3$ in this case. That is, $C_0 \subset CCC$ for any nonscalar conjugacy class C in $\mathrm{GL}_2(F)$ and any conjugacy class C_0 in $\mathrm{GL}_2(F)$ with $\det C_0 = (\det C)^3$. Lemma 6.1 yields $C_0 \subset CCC$ when C_0 is not scalar. Assume now that C_0 is scalar, i.e., $C_0 = \lambda 1_2$ with $0 \neq \lambda \in F$. We have $\det C_0 = \lambda^2 = (\det C)^2$; hence $\det C = \mu^2$ with $\mu = \lambda/\det C$. The inclusion $C_0 \subset CCC$ we want to prove takes the form $1_2 \in C'C'C'$ with $C' = C/\mu$. Since $\det C' = 1$, we have $C'^{-1} = C'$, so we have to prove that $C' \subset C'C'$.

Set $t = \mathrm{tr}\, C'$. If $t = \pm 2$, then $C' \subset C'C'$ by Lemma 5.1. Assume now that $t \neq \pm 2$. By Lemma 6.6, we have to prove that the equation $N_t(x, y) = x^2 - txy + y^2 = -(t-2)(t^2 - t - 2) = -(t-2)^2(t+1)$ has a solution $x, y \in F$. We can set $x = x_0(t-2)$ and $y = y_0(t-2)$.

Conversely, assume now that $\mathrm{cn}(\mathrm{GL}_2(F)) \leqslant 3$. We have to prove that the equation $x^2 - txy + y^2 = -t - 1$ has a solution $x_0, y_0 \in F$ for every $t \neq 2$ in F. When $t = -2$, the equation has solution $x = 1$, $y = 0$. Assume now that $t \neq \pm 2$. Let C be a nonscalar conjugacy class in $\mathrm{GL}_2(F)$ with $\det C = 1$ and $\mathrm{tr}\, C = -t \neq \pm 2$. Since $\mathrm{cn}(\mathrm{GL}_2(F)) \leqslant 3$, CC contains C. By Lemma 6.6, the equation

$$x^2 + txy + y^2 = -(t_0 - 2)(t^2 - t_0 - 2)$$

has a solution $x_1, y_1 \in F$ when $t_0 = t$. So

$$x_1^2 + tx_1y_1 + y^2 = -(t_0 - 2)(t^2 - t_0 - 2) = -(t-2)^2(t+1);$$

hence $x_0^2 - tx_0y_0 + y_0^2 = -t - 1$ for $x_0 = x_1/(t-2)$, $y_0 = -x_1/(t-2)$.

Thus, $\mathrm{cn}(\mathrm{GL}_2(F)) \leqslant 3$ if and only if $x^2 - txy + y^2 = -t - 1$ has a solution $x_0, y_0 \in F$ for every $t \in F$. Combining this with Lemma 6.1 and Corollary 6.5, we obtain Theorem 2.5(a).

(b): We assume first that for every $t \in F$ either $1 - t$ or $1 + t$ has the form $N_t(x, y)$ and prove that $\mathrm{cn}(\mathrm{PGL}_2(F)) \leqslant 3$. That is, $C_0 \subset CCC \cup -CCC$ for any nonscalar conjugacy class C in $\mathrm{GL}_2(F)$ and any conjugacy class C_0 in $\mathrm{GL}_2(F)$ with $\det C_0 = (\det C)^3$. Lemma 6.1 yields $C_0 \subset CCC$ when C_0 is not scalar. Assume now that C_0 is scalar, i.e., $C_0 = \lambda 1_2$ with $0 \neq \lambda \in F$. We have $\det C_0 = \lambda^2 = (\det C)^3$; hence $\det C = \mu^2$ with $\mu = \lambda/\det C$. The inclusion $C_0 \subset CCC \cup -CCC$ we want to prove takes the form $C' \subset C'C' \cup -C'C'$ with $C' = C/\mu$. When $\mathrm{tr}\, C' = 2$, $C' \subset C'C'$ because $\mathrm{card}(F) \geqslant 3$. When $\mathrm{tr}\, C' = -2$, $C' \subset -C'C'$ because $\mathrm{card}(F) \geqslant 3$. Assume now that $t = \mathrm{tr}\, C' \neq \pm 2$. When $-t - 1$ has the form $N_t(x, y)$, $C' \subset C'C'$ by Lemma 6.6 with $t_0 = t$. When $t - 1$ has the form $N_t(x, y)$, we have $-C' \subset C'C'$, i.e., $C' \subset -C'C'$ by Lemma 6.6 with $t_0 = -t$.

Conversely, if $\mathrm{cn}(\mathrm{PGL}_2(F)) \leqslant 3$, then Lemma 6.6 gives that either $-1 - t$ or $-1 + t$ has the form $N_t(x, y)$. Thus, $\mathrm{cn}(\mathrm{PGL}_2(F)) \leqslant 3$ if and only if for every $t \in F$ either $-1 - t$ or $-1 + t$ has the form $N_t(x, y)$. Combining this with Lemma 6.1 and Corollary 6.5, we obtain Theorem 2.5(b).

To prove Theorem 2.5(c), we will use the following refinement of Lemma 6.6.

PROPOSITION 9.1. *Let C_1, C_2, C_3 be nonscalar conjugacy classes in $\mathrm{SL}_2(F)$. Then the equation $g_1 g_2 g_3 = 1_2$ has a solution $g_j \in C_j$ if and only if there are $x_j \in \chi(C_j)$ such that $(x_1, x_2, x_3) \neq (0, 0, 0)$ and*

$$x_1^2 + x_2^2 + x_3^2 + t_3 x_1 x_2 + t_2 x_1 x_3 + t_1 x_2 x_3 = 0, \qquad \text{where} \quad t_j = \mathrm{tr}\, C_j.$$

Proof. Assume first that there are $x_j \in \chi(C_j)$ such that

$$x_1^2 + x_2^2 + x_3^2 + t_3 x_1 x_2 + t_2 x_1 x_3 + t_1 x_2 x_3 = 0.$$

If two of the x_j vanish, then all three are 0. So we can assume that at most one of the x_j vanishes, say, $x_1 x_2 \neq 0$. Set

$$g_1 = \begin{pmatrix} t_1 + \dfrac{x_3}{x_2} & -\dfrac{1}{x_1} - \dfrac{x_3(t_1 x_2 + x_3)}{x_1 x_2^2} \\ x_1 & -\dfrac{x_3}{x_2} \end{pmatrix} \in C_1 \quad \text{and}$$

$$g_2 = \begin{pmatrix} 0 & -\dfrac{1}{x_2} \\ x_2 & t_2 \end{pmatrix} \in C_2.$$

Then

$$\operatorname{tr} g_1 g_2 = t_3 \quad \text{and} \quad g_1 g_2 = \begin{pmatrix} * & * \\ x_3 & * \end{pmatrix},$$

so $g_1 g_2 \in C_3^{-1}$.

Assume now that $g_1 g_2 g_3 = 1_2$ with $g_j \in C_j$. Without loss of generality, we can assume that $(g_2)_{1,1} = 0$. Set $x_j = (g_j)_{2,1} \in \chi(C_j)$. If $x_1 = 0$, then $x_3 \neq 0$, and replacing g_1, g_2, g_3 by $g_3, g_2, g_2^{-1} g_1 g_2$, we can assume that $x_1 \neq 0$. We write

$$g_1 = \begin{pmatrix} t_1 - x & \dfrac{x(t_1 - x) - 1}{x_1} \\ x_1 & x \end{pmatrix} \in C_2 \quad \text{and} \quad g_2 = \begin{pmatrix} 0 & -\dfrac{1}{x_2} \\ x_2 & t_2 \end{pmatrix},$$

so $g_1 g_2 \in C_3^{-1}$.

Since $(g_1 g_2)_{2,1} = (g_3^{-1})_{2,1} = -x_3$, it follows that $x = -x_3/x_2$. Writing that $\operatorname{tr} g_1 g_2 = \operatorname{tr} g_3^{-1} = t_3$, we obtain that

$$x_1^2 + x_2^2 + x_3^2 + t_3 x_1 x_2 + t_2 x_1 x_3 + t_1 x_2 x_3 = 0. \qquad \blacksquare$$

(c): By Lemma 5.1 (when C has an eigenvalue in F) and Corollary 6.5, $3 \leqslant \operatorname{cn}(\mathrm{PSL}_2(F))$. Let us prove that $\operatorname{cn}(\mathrm{PSL}_2(F)) \leqslant 4$. Let C be a nonscalar conjugacy class in $\mathrm{SL}_2(F)$, and C_0 a conjugacy class in $\mathrm{SL}_2(F)$ with $\det C = (\det C)^4$. We have to prove that C_0 or $-C_0 \subset CCCC$. If C_0 is not scalar, then $C_0 \subset CCCC$ by Lemma 7.2.

By Lemma 7.1, $\operatorname{diag}(\lambda, 1/\lambda) \in CC$ for some $\lambda \in F$. Since $\operatorname{diag}(\lambda, 1/\lambda)$ is similar to its inverse in $\mathrm{SL}_2(F)$,

$$1_2 = \operatorname{diag}(\lambda, \lambda) \operatorname{diag}(\lambda, \lambda)^{-1} \in CCCC.$$

Thus, $\operatorname{cn}(\mathrm{PSL}_2(F)) \leqslant 4$. Assume now that $\operatorname{cn}(\mathrm{PSL}_2(F)) = 3$. Then 1_2 or $-1_2 \in CCC$, i.e., C^{-1} or $-C^{-1} \subset CC$, for any nonscalar conjugacy class C in $\mathrm{SL}_2(F)$. When $t \neq \pm 2$, by Lemma 6.6, we obtain that $x^2 - txy + y^2 = -(t_0 - 2)(t^2 - t_0 - 2)$ for some $x, y \in F$, where $t_0 = t$ or $-t$. When $t_0 = t$ we have

$$-(t_0 - 2)(t^2 - t_0 - 2) = -(t + 1)(t - 2)^2,$$

so $-(t+1)$ has the form $N_t(x', y')$. When $t_0 = -t$, we have

$$-(t_0 - 2)(t^2 - t_0 - 2) = (t^2 - 4)(t - 1);$$

hence $t - 1$ has the form $N_t(x', y')$ (because both $t + 2$ and $t - 2$ have such a form).

When $t = \pm 2$, then $1 = -1 - t$ or $-1 + t$ has the form $N_t(x', y')$. So for every element $t \in F$ either $-1 - t$ or $-1 + t$ has the form $N_t(x', y')$. In particular, for $t = 0$, -1 is the sum of two squares in F.

We have to prove that every element c_0 of the field F is the sum of two squares. If -1 is a square in F, this is clearly true. So we assume that -1 is not a square. Since -1 is the sum of two squares, c_0 is such a sum if and only if $-c_0$ is.

We take an arbitrary $c_0 \in F$. Let C, C_0 be the conjugacy classes in $\mathrm{SL}_2(F)$ with $\operatorname{tr} C = \operatorname{tr} C_0 = 0$, $\chi(C) \ni 1$, $\chi(C_0) \ni c_0$.

Since $\operatorname{cn}(\mathrm{PSL}_2(F)) \leqslant 3$, we have $g_1 g_2 g_3 = g_0$ with $g_1, g_2, g_3 \in C$ and $g_0 \in C_0 \cup -C_0$. If $g_1 g_2$ is scalar, then the equality $g_0 = \pm g_3$ gives that c_0 is the sum of two squares. Otherwise, we apply Theorem 2.6 to $g_1 g_2 = g_0 g_3^{-1}$ to obtain that c_0 is the sum of two squares.

(d): By Lemma 7.2, $\operatorname{cn}(\mathrm{SL}_2(F)) \leqslant \operatorname{ecn}(\mathrm{SL}_2(F)) \leqslant 5$. Let us show that $4 \leqslant \operatorname{cn}(\mathrm{SL}_2(F))$ when F is not a $C_1(2)$-field. We pick $t, c_0 \in F$ such that $t \neq \pm 2$ and $-c_0$ is not of the form $N_t(x, y)$. Let C, C_0 be the conjugacy classes in $\mathrm{SL}_2(F)$ with $\operatorname{tr} C = t = \operatorname{tr} C_0$, $\chi(C) \ni 1$, and $\chi(C_0) \ni c_0$. We will show that C_0 is not contained in CCC. Otherwise, we have

$$g_1 g_2 = g_3^{-1} g_0 \qquad \text{with} \quad g_1, g_2, g_3 \in C \text{ and } g_0 \in C_0.$$

By Theorem 2.6,

$$h g_1 h^{-1} = g_3^{-1} \text{ and } h g_2 h^{-1} = g_0 \qquad \text{for some} \quad h \in \mathrm{GL}_2(F).$$

Then

$$\chi(C) = \chi(g_1) = (\det h)\chi(g_3^{-1}) = -(\det h)\chi(C)$$

and

$$\chi(C) = \chi(g_2) = (\det h)\chi(g_0) = (\det h) c_0 \chi(C);$$

hence $-c_0 \in \chi(C)$, which is a contradiction.

Thus, $4 \leqslant \mathrm{cn}(\mathrm{SL}_2(F)) \leqslant 5$. Assume now that $\mathrm{cn}(\mathrm{SL}_2(F)) = 4$. We want to prove that -1 is the sum of four squares. Assume that -1 is not a square in F (otherwise, -1 is the sum of four squares and we are done). Let C be the conjugacy class of h_0 in $\mathrm{SL}_2(F)$ [i.e., $\mathrm{tr}\, C = 0$ and $\chi(C)$ consists of all nonzero sums of two squares].

We claim that $\mathrm{tr}\, g_0$ is the sum of four squares for any $g_0 \in -CC$. Indeed, let $g_0 = -g_1 g_2$ with $g_1, g_2 \in C$. To compute $\mathrm{tr}\, g_0$, we can assume without loss of generality that

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We write

$$g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\mathrm{tr}\, g_0 = c - b$. Since $b, c \in \chi(C)$ are each the sum of two squares, $\mathrm{tr}\, g_0$ is the sum of four squares.

Now we write $-1_2 = g_1 g_2 g_3 g_4$ with $g_j \in C$. Set $g_0 = -g_1 g_2 = g_4^{-1} g_3^{-1}$. As shown above, $\mathrm{tr}\, g_0$ is the sum of two squares. On the other hand $g_4^{-1} g_3^{-1}$ is similar in $\mathrm{GL}_2(F)$ to $g_3 g_4 \in CC$, so $-\mathrm{tr}\, g_0$ is also the sum of four squares. Thus, -1 is the sum of four squares.

REMARK 9.2. Using Proposition 9.1, one easily gets the following necessary and sufficient conditions on F for $\mathrm{cn}(\mathrm{PSL}_2(F)) \leqslant 3$:

(1) for any $t \in F$ there are $\varepsilon \in F$ and x_j in the range of εN_t such that $\varepsilon^2 = 1$, $(x_1, x_2, x_3) \neq (0, 0, 0)$, and $x_1^2 + x_2^2 + x_3^2 + \varepsilon t x_1 x_2 + \varepsilon t x_1 x_3 + \varepsilon t x_2 x_3 = 0$;

(2) for any $t, t_0, c_0 \in F$ there are $\varepsilon, t_4, c_4, x_j, y_j \in F$ such that $\varepsilon^2 = 1$;

$$x_1^2 + x_2^2 + x_3^2 + t_4 x_1 x_2 + t x_1 x_3 + t x_2 x_3 = 0;$$

$$y_1^2 + y_2^2 + y_3^2 + t y_1 y_2 + \varepsilon t_0 y_1 y_3 + t_4 y_2 y_3 = 0;$$

$(x_1, x_2, x_3) \neq (0, 0, 0) \neq (y_1, y_2, y_3)$; $x_1, x_2, -y_2$ are in the range of N_t; x_3, y_1 are in the range of $c_4 N_{t_1}$; and y_2 is in the range of $\varepsilon c_0 N_{t_0}$.

The first condition is obtained from $\varepsilon 1_2 \in CCC$ with $t = \mathrm{tr}\, C$. The second condition is obtained by rewriting $g_1 g_2 g_3 = g_0$, where $g_1, g_2, g_3 \in C$, $g_0 \in \varepsilon C_0$, $1 \in \chi(C)$ as $g_1 g_2 g_4 = 1_2 = g_4 g_0 g_3^{-1}$ with $g_4 = g_1 g_2$, $t_4 = \mathrm{tr}\, g_4$, and $c_4 \in \chi(g_4)$. However, these conditions seem to be too complicated, and we could not use them to get new insights.

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Received 5 July 1993; final manuscript accepted 27 December 1993